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Approximation of functions with linear positive operators which fix $\{1, \varphi\}$ and $\{1, \varphi^2\}$

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Abstract

In this manuscript, linear and positive operators described on bounded and unbounded intervals that fix the function sets $\{1, \varphi\}$ and $\{1, \varphi^2\}$ such that $\varphi \in C[0, 1]$ are presented. Then we present different types of operators by choosing different functions and values. Finally, Voronovskaya type theorems are given for this newly defined sequences of linear and positive operators.

1 Introduction & preliminaries

The most important turning point in approximation theory is undoubtedly that Weierstrass proved the famous theorem in 1885 on the approximate of continuous functions in polynomials. In the following years, a number of different proofs of this theorem have been made. One of the most interesting of these is the proof of S. Bernstein in 1912, [5]. In 1953, the proof of S. Bernstein was further simplified in the introduction to the concept of "Linear Positive Operators" that belongs to Korovkin. Thereupon, studies have increased on linear and positive operators, which are the approximation of continuous functions such as Bernstein, Szász-Mirakyan, and Baskakov operators.

Besides, King's brilliant idea [7] make a tremendous impact on the approximation theory and has been successfully implemented to several well-recognized sequences of operators. The main motivation of King is fixing the function x^2 instead of function x for the classical Bernstein operators which

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approximate better compared to the past. In this direction, polynomial type operators preserve the 1 and $t^2 + \alpha t$ introduced in [6]. In this study, Bernstein type operators described by $B_n(f \circ \tau^{-1}) \circ \tau$ were considered such that τ is any infinitely times continuously differentiable function on $[0, 1]$, $\tau(\{0, 1\}) = \{0, 1\}$ and $\tau^{'} > 0$. Then, Durrmeyer-type generalization of referred operators was also examined in [2]. Later on, in [3], [4], the authors presented and studied the similar modification of the sequence of the generalized Szász-Mirakyan operators. Finally, in [1], the authors provided linear and positive operators defined on bounded and unbounded intervals that fix the functions τ and τ^2, where $\tau(\{0, 1\}) = \{0, 1\}$, $\tau \in [0, 1]$ and $\tau' > 0$.

Motivated by [1], we introduce linear and positive operators described on bounded and unbounded intervals which fix the function sets $\{1, \varphi\}$ and $\{1, \varphi^2\}$ such that $\varphi \in C[0, 1]$. Then we provided different types of operators by choosing different functions and values. Finally, Voronovskaya type theorems are given for this newly defined sequences of linear and positive operators.

Throughout this and the next sections, we denote by e_j the polynomial functions described by $e_j(t) = t^j$ for $x \in \mathbb{R}^+$ and $j \in \mathbb{N}$. Moreover, we denote by $C([0, 1]$ and $C([0, \infty[)$ the space of whole real valued continuous maps on $[0, 1]$ and $[0, \infty[$, respectively.

The overall structure of the paper takes the form of five sections including this section. The remainder of the presented work is constituted in five sections. In Section 2, the new sequences of linear positive operators which preserve $\{1, \varphi\}$ are introduced for both bounded and unbounded intervals. In section 3, the similar linear positive operators which fix $\{1, \varphi^2\}$ are presented. Voronovskaya type theorems of these newly defined operators have been presented in Section 4, while some consequences and further directions of study are discussed in Section 5.

2 New generalizations of linear positive operators preserve $\{1, \varphi\}$

2.1 New operators on bounded interval

In this subsection, we introduce the new linear and positive operators defined on a bounded interval which fix the functions 1 and φ.

Let $\mathcal{B}_n : C[0, 1] \to C[0, 1]$ be a sequence of linear and positive operator defined on bounded intervals such that

1. $\mathcal{B}_n e_0 = e_0$,
2. $\mathcal{B}_n e_1 = e_1$,
3. $\mathcal{B}_n e_2 = \alpha_n e_2 + \beta_n e_1$,

where $\lim_{n\to\infty} \alpha_n = 1$ and $\lim_{n\to\infty} \beta_n = 0$, thus the operator $\mathcal{B}_n$ confirms the King's well-known theorem.

In addition to this, let $\varphi : [0,1] \to [\eta, \xi]$ be continuous map such that φ is strictly increasing for $x \in [0,1]$ and $0 < \eta < \xi$. Here it is obvious that $\varphi(0) = \eta$ and $\varphi(1) = \xi$.

Definition 1. *Let $\mathcal{B}_n$ be an operator and φ be a function defined as above. In the circumstance, for each $f \in C[0,1]$ and $x \in [0,1]$, the generalized linear positive operators on bounded intervals are defined by*

$$\Delta_n^{\mathcal{B}}(f;x) = \mathcal{B}_n \left(f \circ \varphi^{-1} \left(\eta + (\xi - \eta)t\right); \frac{\varphi(x) - \eta}{\xi - \eta} \right). \tag{2.1}$$

Lemma 1. *For each $x \in [0,1]$, we have*

1. $\Delta_n^{\mathcal{B}}(e_0(t);x) = e_0(x)$,
2. $\Delta_n^{\mathcal{B}}(\varphi(t);x) = \varphi(x)$,
3. $\Delta_n^{\mathcal{B}}(\varphi^2(t);x) = \alpha_n \varphi^2(x) + (2\eta\varphi(x) - \eta^2)(1 - \alpha_n) + \beta_n(\xi - \eta)(\varphi(x) - \eta)$.

Proof. The proof of this lemma is easily derived by direct computations, which is left to the reader. □

Note that, $\Delta_n^{\mathcal{B}}(\varphi^2(t);x) \to \varphi^2(x)$ in the limit case because $\lim_{n\to\infty} \alpha_n = 1$ and $\lim_{n\to\infty} \beta_n = 0$. Now, we present some examples below by choosing arbitrary $\varphi(x)$, η and ξ.

Bounded Interval				
$\mathcal{B}_n$	$\varphi(x)$	η	ξ	$\Delta_n^{\mathcal{B}}$
Bernstein (B_n)	$1 + x$	1	2	$B_n(f(t);x)$
Bernstein (B_n)	$e^{\mu x}$	1	e^{μ}	$B_n\left(f\left(\frac{1}{\mu}\log(1 + (e^{\mu} - 1)t)\right); \frac{e^{\mu x} - 1}{e^{\mu} - 1}\right)$

Table 1: Some examples of new operators by choosing arbitrary $\varphi(x)$, η and ξ

2.2 New operators on unbounded interval

Similarly, we provide the new linear and positive operators defined on unbounded interval preserve the functions 1 and ϕ.

Let $\mathcal{U}_n : C[0,\infty[\to C[0,\infty[$ be an sequence of linear and positive operators defined on unbounded intervals such that

1. $\mathcal{U}_n e_0 = e_0$,
2. $\mathcal{U}_n e_1 = e_1$,
3. $\mathcal{U}_n e_2 = \gamma_n e_2 + \theta_n e_1$

where $\lim_{n\to\infty} \gamma_n = 1$ and $\lim_{n\to\infty} \theta_n = 0$, thus, the operator $\mathcal{U}_n$ confirms the King's well-known theorem.

Let $\phi : [0, \infty[\to [\eta, \infty[$ be continuous map such that ϕ is strictly increasing for $x \in [0, \infty[$ and $0 < \eta$. Here it is obvious that $\phi(0) = \eta$.

Definition 2. *Let $\mathcal{U}_n$ be an operator and ϕ be a function defined as above. In the given circumstances, for the each $f \in C[0, \infty[$ and $x \in [0, \infty[$, the generalized linear positive operators on unbounded intervals are defined as*

$$\nabla_n^{\mathcal{U}}(f; x) = \mathcal{U}_n \left(f \circ \phi^{-1} (\eta + t) ; \phi(x) - \eta \right). \tag{2.2}$$

Lemma 2. *For each $x \in [0, \infty[$, we have*

1. $\nabla_n^{\mathcal{U}}(e_0(t); x) = e_0(x)$,
2. $\nabla_n^{\mathcal{U}}(\phi(t); x) = \phi(x)$,
3. $\nabla_n^{\mathcal{U}}(\phi^2(t); x) = \gamma_n \phi^2(x) + (2\eta\phi(x) - \eta^2)(1 - \gamma_n) + \beta_n(\phi(x) - \eta)$.

Proof. The proof of this lemma is easily derived by elementary calculus, which is left to the reader. □

Similarly, $\nabla_n^{\mathcal{B}}(\phi^2(t); x) \to \phi^2(x)$ in the limit case since $\lim_{n\to\infty} \gamma_n = 1$ and $\lim_{n\to\infty} \theta_n = 0$. Now we provide some examples below by choosing arbitrary $\phi(x)$ and η.

Unbounded Interval			
$\mathcal{U}_n$	$\phi(x)$	η	$\nabla_n^{\mathcal{U}}$
Szász-Mirakyan (S_n)	$e^{\mu x} + 1$	2	$S_n \left(f \left(\frac{1}{\mu} \log(1 + t) \right) ; e^{\mu x} - 1 \right)$
Baskakov (K_n)	$e^{\mu x}$	1	$K_n \left(f \left(\frac{1}{\mu} \log(t) \right) ; e^{\mu x} - 1 \right)$
Balázs (M_n)	$1 + x$	1	$M_n \left(f \left(t \right) ; x \right)$
BBH (T_n)	x^2	1	$T_n \left(f \left(\sqrt{1 + t} \right) ; x^2 - 1 \right)$

Table 2: Some examples of new operators by choosing arbitrary $\phi(x)$ and η

We see different values $(\Delta_n^{\mathcal{B}})_{n\geq 1}$ and $(\nabla_n^{\mathcal{U}})_{n\geq 1}$ in Tables (1) and (2) for different choices of $\varphi(x)$ and $\phi(x)$, respectively. As a result of the above consequences, we can argue that $(\Delta_n^{\mathcal{B}})_{n\geq 1}$ and $(\nabla_n^{\mathcal{U}})_{n\geq 1}$ preserve the Korovkin test functions which means they are approximation processes on any compact interval $K \subset \mathbb{R}^+$.

3 New generalizations of linear positive operators preserve $\{1, \varphi^2\}$

3.1 New operators on bounded interval

Now, in this subsection, we introduce the new linear and positive operators defined on bounded interval which fix the functions 1 and φ^2.

Definition 3. *Let $\mathcal{B}_n$ be an operator and φ be a function defined as above. In the given circumstances, for the each $f \in C[0,1]$ and $x \in [0,1]$, the generalized linear positive operators on bounded intervals are defined as*

$$\Delta_n^{\mathcal{B}}(f;x) = \mathcal{B}_n\left(f \circ \varphi^{-1}\left(\sqrt{\eta + (\xi-\eta)t}\right); \frac{\varphi^2(x)-\eta}{\xi-\eta}\right). \tag{3.1}$$

Lemma 3. *For each $x \in [0,1]$, we get*

1. $\Delta_n^{\mathcal{B}}(e_0(t);x) = e_0(x)$,
2. $\Delta_n^{\mathcal{B}}(\varphi^2(t);x) = \varphi^2(x)$.

Proof. The proof of this lemma can be obtain easily with the help of fundamental calculus. $\square$

Bounded Interval				
$\mathcal{B}_n$	$\varphi(x)$	η	ξ	$\Delta_n^{\mathcal{B}}$
Bernstein (B_n)	$1+x$	1	2	$B_n(\sqrt{f(t)}; x^2+2x)$
Bernstein (B_n)	$e^{\mu x}$	1	e^{μ}	$B_n\left(f\left(\frac{1}{\mu}\log(\sqrt{1+(e^{\mu}-1)t})\right); \frac{e^{2\mu x}-1}{e^{\mu}-1}\right)$

Table 3: Some examples of new operators by choosing arbitrary $\varphi(x)$, η and ξ

3.2 New operators on unbounded interval

Similarly, we provide a new linear and positive operators defined on unbounded interval preserve the functions 1 and ϕ^2.

Definition 4. *Let $\mathcal{U}_n$ be an operator and ϕ be a function defined as above. In the given circumstances, for the each $f \in C[0,\infty[$ and $x \in [0,\infty[$, the generalized linear positive operators on unbounded interval define as*

$$\nabla_n^{\mathcal{U}}(f;x) = \mathcal{U}_n\left(f \circ \phi^{-1}\left(\sqrt{\eta + t}\right); \phi^2(x) - \eta\right). \tag{3.2}$$

Lemma 4. *For each $x \in [0,\infty[$, we have*

1. $\nabla_n^{\mathcal{U}}(e_0(t);x) = e_0(x)$,
2. $\nabla_n^{\mathcal{U}}(\phi^2(t);x) = \phi^2(x)$.

Proof. The proof of this lemma can be also obtain easily with the help of fundamental calculus. □

Unbounded Interval			
$\mathcal{U}_n$	$\phi(x)$	η	$\nabla_n^{\mathcal{U}}$
Szász-Mirakyan (S_n)	$e^{\mu x}+1$	2	$S_n\left(f\left(\sqrt{\frac{1}{\mu}\log(1+t)}\right); e^{2\mu x}-1\right)$
Baskakov (K_n)	$e^{\mu x}$	1	$K_n\left(f\left(\sqrt{\frac{1}{\mu}\log(t)}\right); e^{2\mu x}-1\right)$
Balázs (M_n)	$1+x$	1	$M_n\left(f\left(\sqrt{1+t}-1\right); x^2+x\right)$
BBH (T_n)	x^2	1	$T_n\left(f\left((1+t)^{1/4}\right); x^4-1\right)$

Table 4: Some examples of new operators by choosing arbitrary $\phi(x)$ and η

Similarly, we can obtain different operators for different functions, see in Tables (3) and (4).

4 Voronovskaya type theorems of $(\Delta_n^{\mathcal{B}})_{n\geq 1}$ and $(\nabla_n^{\mathcal{U}})_{n\geq 1}$

Now, lastly, we provide the Voronovskaya type theorems for the introduced new sequences of linear positive operators preserve $\{1,\varphi\}$ and $\{1,\varphi^2\}$.

4.1 Theorems of operators preserve $\{1,\varphi\}$

Theorem 1. *Let $\mathcal{B}_n : C[0,1] \to C[0,1]$ be an linear positive operator on bounded interval described as above and $\Delta_n^{\mathcal{B}}$ is introduced in (2.1). Let $f \in C[0,1]$ with $f^{''}(t)$ for finite $t \in [0,1]$ and there exist $\lambda \in C[0,1]$, such that*

$$\lim_{n\to\infty} n(\mathcal{B}_n f(t) - f(t)) = \lambda(t)\|f^{''}\|,$$

in the given circumstances, we conclude

$$\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) = (\xi-\eta)^2 \lambda\left(\frac{\varphi(x)-\eta}{\xi-\eta}\right) \frac{\|f^{''}\|\varphi^{'}(x) - \|f^{'}\|\varphi^{''}(x)}{[\varphi^{'}(x)]^3}.$$

Proof. First of all, we need to find $n(\Delta_n^{\mathcal{B}} f(x) - f(x))$, which is

$$\begin{aligned} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) &= n\mathcal{B}_n\left(f\circ\varphi^{-1}\left(\eta+(\xi-\eta)t\right); \frac{\varphi(x)-\eta}{\xi-\eta}\right) - f(x), \\ &= n\mathcal{B}_n\left(f\circ\varphi^{-1}\left(\eta+(\xi-\eta)t\right); \frac{\varphi(x)-\eta}{\xi-\eta}\right) \\ &\quad - f\circ\varphi^{-1}(\eta+(\xi-\eta)t)\Big|_{t=\frac{\varphi(x)-\eta}{\xi-\eta}}. \end{aligned}$$

Since

$$\lim_{n\to\infty} n(\mathcal{B}_n f(t) - f(t)) = \lambda(t) f^{''}(t),$$

from the previously proved results, we conclude that

$$\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) = \lambda\left(\frac{\varphi(x)-\eta}{\xi-\eta}\right) \frac{d^2}{du^2}\left(f\circ\varphi^{-1}\left(\eta+(\xi-\eta)u\right)\right)\Big|_{u=\frac{\varphi(x)-\eta}{\xi-\eta}}.$$

Here, $\dfrac{du}{dx} = \dfrac{\varphi^{'}(x)}{\xi-\eta}$ as $u = \dfrac{\varphi(x)-\eta}{\xi-\eta}$. Now, we find the second derivative of the function stated above, as;

$$\begin{aligned} &\frac{d^2}{du^2}\left(f\circ\varphi^{-1}\left(\eta+(\xi-\eta)u\right)\right)\Big|_{u=\frac{\varphi(x)-\eta}{\xi-\eta}} = \\ &\qquad = \frac{d}{du}\left(\frac{d}{du}\left(f\circ\varphi^{-1}\left(\eta+(\xi-\eta)u\right)\right)\Big|_{u=\frac{\varphi(x)-\eta}{\xi-\eta}}\right), \\ &\qquad = \frac{d}{du}\left(\frac{dx}{du}\frac{d}{dx}f(x)\right), \\ &\qquad = (\xi-\eta)\frac{d}{du}\left(\frac{f^{'}(x)}{\varphi^{'}(x)}\right), \\ &\qquad = (\xi-\eta)\frac{dx}{du}\frac{d}{dx}\left(\frac{f^{'}(x)}{\varphi^{'}(x)}\right), \\ &\qquad = (\xi-\eta)^2\frac{f^{''}(x)\varphi^{'}(x) - f^{'}(x)\varphi^{''}(x)}{[\varphi^{'}(x)]^3}. \end{aligned}$$

So, we deduce that

$$\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) = (\xi-\eta)^2 \lambda\left(\frac{\varphi(x)-\eta}{\xi-\eta}\right) \frac{f^{''}(x)\varphi^{'}(x) - f^{'}(x)\varphi^{''}(x)}{[\varphi^{'}(x)]^3}.$$

Thus the proof is completed. □

Corollary 1. *Let $\mathcal{B}_n$ be classical sequence of Bernstein operators and $\varphi(x) = 1+x$. So for $\eta = 1$ and $\xi = 2$, we get*

$$\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) = \lambda\,(x)\,\|f^{''}\|,$$

where $\lambda(x)$ is a function defined in Voronovskaya theorem of classical Bernstein operators.

Theorem 2. *Let $\mathcal{U}_n : C[0,\infty[\to C[0,\infty[$ be an linear positive operator on unbounded interval described as above and $\nabla_n^{\mathcal{B}}$ is introduced in (2.2). Let $f \in C[0,\infty[$ with $f^{''}(t)$ for finite $t \in [0,\infty[$ and there exists $\vartheta \in C[0,\infty[$, such that*

$$\lim_{n\to\infty} n(\mathcal{U}_n f(t) - f(t)) = \vartheta(t)\|f^{''}\|,$$

then we have

$$\lim_{n\to\infty} n(\nabla_n^{\mathcal{U}} f(x) - f(x)) = \vartheta\,(\phi(x)-\eta)\,\frac{\|f^{''}\|\phi^{'}(x) - \|f^{'}\|\phi^{''}(x)}{[\phi^{'}(x)]^3}.$$

Proof. The similar proof for this theorem can be deduced by following the similar steps of proof of Theorem 1. □

Corollary 2. *Let $\mathcal{U}_n$ be classical sequence of Szász-Mirakyan operators and $\phi(x) = e^{\mu x}+1$. So for $\eta = 2$, we get*

$$\lim_{n\to\infty} n(\nabla_n^{\mathcal{U}} f(x) - f(x)) = \vartheta\,(e^{\mu x}-1)\,\frac{\|f^{''}\| - \mu\|f^{'}\|}{\mu^2 e^{2\mu x}},$$

where $\vartheta(x)$ is a function defined in Voronovskaya theorem of classical Szász-Mirakyan operators.

4.2 Theorems of operators preserve $\{1,\varphi^2\}$

Theorem 3. *Let $\mathcal{B}_n : C[0,1] \to C[0,1]$ be linear positive operators on bounded interval described as above and $\Delta_n^{\mathcal{B}}$ is introduced in (3.1). Let $f \in C[0,1]$ with $f^{''}(t)$ for finite $t \in [0,1]$ and there exist $\lambda \in C[0,1]$, such that*

$$\lim_{n\to\infty} n(\mathcal{B}_n f(t) - f(t)) = \lambda(t)\|f^{''}\|,$$

then we have

$$\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x))$$
$$= (\xi-\eta)^2\lambda\left(\frac{\varphi^2(x)-\eta}{\xi-\eta}\right)\frac{\|f^{''}\|\varphi(x)\varphi^{'}(x) - \|f^{'}\|([\varphi^{'}(x)]^2 + \varphi(x)\varphi^{''}(x))}{4[\varphi(x)\varphi^{'}(x)]^3}.$$

Proof. Initially, we need to find $n(\Delta_n^{\mathcal{B}} f(x) - f(x))$, as follows;

$$\begin{aligned} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) &= n\mathcal{B}_n \left(f \circ \varphi^{-1} \left(\sqrt{\eta + (\xi-\eta)t} \right); \frac{\varphi^2(x)-\eta}{\xi-\eta} \right) - f(x), \\ &= n\mathcal{B}_n \left(f \circ \varphi^{-1} \left(\eta + (\sqrt{\xi-\eta)t} \right); \frac{\varphi^2(x)-\eta}{\xi-\eta} \right) \\ &- f \circ \varphi^{-1}(\sqrt{\eta+(\xi-\eta)t})\big|_{t=\frac{\varphi^2(x)-\eta}{\xi-\eta}}. \end{aligned}$$

Since

$$\lim_{n\to\infty} n(\mathcal{B}_n f(t) - f(t)) = \lambda(t) f^{''}(t),$$

from the previously proved results, we deduce that,

$$\begin{aligned} &\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) \\ &= \lambda\left(\frac{\varphi^2(x)-\eta}{\xi-\eta}\right) \frac{d^2}{du^2}\left(f\circ\varphi^{-1}\left(\sqrt{\eta+(\xi-\eta)u}\right)\right)\big|_{u=\frac{\varphi^2(x)-\eta}{\xi-\eta}}. \end{aligned}$$

Here, $\dfrac{du}{dx} = \dfrac{2\varphi(x)\varphi^{'}(x)}{\xi-\eta}$ as $u = \dfrac{\varphi^2(x)-\eta}{\xi-\eta}$. Now we need to find the second derivative of the aforementioned functions as follows;

$$\begin{aligned} &\frac{d^2}{du^2}\left(f\circ\varphi^{-1}\left(\sqrt{\eta+(\xi-\eta)u}\right)\right)\big|_{u=\frac{\varphi^2(x)-\eta}{\xi-\eta}} \\ &= \frac{d}{du}\left(\frac{d}{du}\left(f\circ\varphi^{-1}\left(\sqrt{\eta+(\xi-\eta)u}\right)\right)\big|_{u=\frac{\varphi^2(x)-\eta}{\xi-\eta}} \right), \\ &= \frac{d}{du}\left(\frac{dx}{du}\frac{d}{dx} f(x) \right), \\ &= (\xi-\eta)\frac{d}{du}\left(\frac{f^{'}(x)}{2\varphi(x)\varphi^{'}(x)} \right), \\ &= (\xi-\eta)\frac{dx}{du}\frac{d}{dx}\left(\frac{f^{'}(x)}{2\varphi(x)\varphi^{'}(x)} \right), \\ &= (\xi-\eta)^2 \frac{f^{''}(x)\varphi(x)\varphi^{'}(x) - f^{'}(x)([\varphi^{'}(x)]^2 + \varphi(x)\varphi^{''}(x))}{4[\varphi(x)\varphi^{'}(x)]^3}. \end{aligned}$$

So we have that,

$$\begin{aligned} &\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) = \\ &(\xi-\eta)^2 \lambda\left(\frac{\varphi^2(x)-\eta}{\xi-\eta}\right) \frac{f^{''}(x)\varphi(x)\varphi^{'}(x) - f^{'}(x)([\varphi^{'}(x)]^2 + \varphi(x)\varphi^{''}(x))}{4[\varphi(x)\varphi^{'}(x)]^3}. \end{aligned}$$

Hence the desired result is obtained.

□

Corollary 3. *Let $\mathcal{B}_n$ be classical sequence of Bernstein operators and $\varphi(x) = e^{\mu x}$. So for $\eta = 1$ and $\xi = e^{\mu}$, we get*

$$\lim_{n\to\infty} n(\Delta_n^{\mathcal{B}} f(x) - f(x)) = (e^{\mu-1})^2 \lambda \left(\frac{e^{2\mu x}-1}{e^{\mu}-1}\right) \frac{\|f^{''}\| - 2\mu\|f^{'}\|}{4\mu^2 e^{4\mu x}},$$

where $\lambda(x)$ is a function defined in Voronovskaya theorem of classical Bernstein operators.

Theorem 4. *Let $\mathcal{U}_n : C[0,\infty[\to C[0,\infty[$ be linear positive operators on unbounded interval described as above and $\nabla_n^{\mathcal{B}}$ be introduced in (3.2). Let $f \in C[0,\infty[$ with $f^{''}(t)$ for finite $t \in [0,\infty[$ and there exst $\vartheta \in C[0,\infty[$, such that*

$$\lim_{n\to\infty} n(\mathcal{U}_n f(t) - f(t)) = \vartheta(t)\|f^{''}\|,$$

then we have

$$\lim_{n\to\infty} n(\nabla_n^{\mathcal{U}} f(x) - f(x))$$
$$= \vartheta\left(\phi^2(x) - \eta\right) \frac{\|f^{''}\|\phi(x)\phi^{'}(x) - \|f^{'}\|([\phi^{'}(x)]^2 + \phi(x)\phi^{''}(x))}{4[\phi(x)\phi^{'}(x)]^3}.$$

Proof. Similarly the proof of this theorem can be concluded by following the similar steps of the proof of Theorem 3. □

Corollary 4. *Let $\mathcal{U}_n$ be classical sequence of Baskakov operators and $\phi(x) = e^{\mu x} + 1$. So for $\eta = 1$, we get*

$$\lim_{n\to\infty} n(\nabla_n^{\mathcal{U}} f(x) - f(x)) \quad = \quad \vartheta(x) \frac{(1+x)\|f^{''}\| - \|f^{'}\|}{4(1+x)^3},$$

where $\vartheta(x)$ is a function defined in Voronovskaya theorem of classical Baskakov operators.

5 Concluding remarks

In this study, we provide linear and positive operators described on bounded and unbounded intervals which fix the function sets $\{1,\varphi\}$ and $\{1,\varphi^2\}$ such that $\varphi \in C[0,1]$. Then we present different type of operators by choosing different functions and values. Finally, Voronovskaya type theorems are given for this newly defined sequences of linear and positive operators.

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